Complete State from the Triangle Inequality for Quaternions:

Joy Christian

Recall that a parallelized 3-sphere, $S^3$ [or the group SU(2)], is a set of unit quaternions, defined as

$$S^3 := \left\{ \mathbf{q}(\phi_o, I \cdot \mathbf{v}_o) = \exp[(I \cdot \mathbf{v}_o) \phi_o] \mid \|\mathbf{q}(\phi_o, I \cdot \mathbf{v}_o)\| = 1 \right\},$$

(1)

where $I \cdot \mathbf{v}_o$ is an initial bivector rotating about $\mathbf{v}_o \in \mathbb{R}^3$ and $\phi_o$ is half of its initial rotation angle.

Using the notation of geometric product, from Eq. (2), and for $\|\|$ such that

$$\mathbf{p}_o(\eta_{ne_o}, \mathbf{n} \wedge \mathbf{e}_o) := \cos(\eta_{ne_o}) + \frac{\mathbf{n} \wedge \mathbf{e}_o}{\|\mathbf{n} \wedge \mathbf{e}_o\|} \sin(\eta_{ne_o})$$

(2)

and

$$\mathbf{q}_o(\eta_{zs_o}, \mathbf{z} \wedge \mathbf{s}_o) := \cos(\eta_{zs_o}) + \frac{\mathbf{z} \wedge \mathbf{s}_o}{\|\mathbf{z} \wedge \mathbf{s}_o\|} \sin(\eta_{zs_o}).$$

(3)

Note that, although $\mathbf{p}_o$ and $\mathbf{q}_o$ are normalized to unity, their sum $\mathbf{p}_o + \mathbf{q}_o$ need not be. In fact, from the triangle inequality (which holds for any arbitrary pair of quaternions) we have the inequality

$$\|\mathbf{p}_o\| + \|\mathbf{q}_o\| \geq \|\mathbf{p}_o + \mathbf{q}_o\|.$$

(4)

Multiplying with $\|\mathbf{p}_o\| = 1$ on both sides, and simplifying a bit, reduces this triangle inequality to

$$\|\mathbf{p}_o\|^2 \geq \|\mathbf{p}_o + \mathbf{q}_o\| - 1.$$

(5)

This inequality allows us to make the following choice for the set of complete states for our system:

$$\Lambda := \left\{ (\mathbf{p}_o, \mathbf{q}_o) \mid \|\mathbf{p}_o + \mathbf{q}_o\| = 1 + \sin^2(\eta_{ne_o}) + f^2(\eta_{zs_o}) \quad \forall \mathbf{n} \in T_pS^3 \right\},$$

(6)

where $f(\eta_{zs_o})$ is an arbitrary function of $\eta_{zs_o}$, which, as we shall soon see, satisfies the condition

$$f(\eta_{zs_o}) \leq \|\cos(\eta_{ne_o})\|.$$

(7)

Thus the choice (6) respects the condition $\|\mathbf{p}_o + \mathbf{q}_o\| \leq 2$ following from Eq. (4). Substituting for

$$\|\mathbf{p}_o\|^2 = \cos^2(\eta_{ne_o}) + \sin^2(\eta_{ne_o})$$

(8)

from Eq. (2), and for $\|\mathbf{p}_o + \mathbf{q}_o\|$, with $f(\eta_{zs_o}) = -1 + \frac{2}{\sqrt{3 + (\frac{\eta_{zs_o}}{\pi})}}$ from Eq. (6), into Eq. (5) leads to

$$\left. \right| \cos(\eta_{ne_o}) \right| \geq -1 + \frac{2}{\sqrt{1 + 3 \left( \frac{\eta_{zs_o}}{\pi} \right)}}$$

(9)

Without loss of generality we can now identify $\mathbf{e}_o \in \mathbb{R}^3$ as a random vector and $\eta_{zs_o} \equiv \eta_o \in [0, \pi]$ as a random scalar. This finally allows us to rewrite the set (6) of complete states of the system as

$$\Lambda := \left\{ (\mathbf{p}_o, \mathbf{q}_o) \mid \cos(\eta_{ne_o}) \geq -1 + \frac{2}{\sqrt{1 + 3 \left( \frac{\eta_{zs_o}}{\pi} \right)}} \quad \forall \mathbf{n} \in T_pS^3 \right\},$$

(10)

which, for $\mathbf{n} = \mathbf{a}$, corresponds to measurement results of the form

$$\pm 1 = \mathcal{A}(\mathbf{a}; \mathbf{e}_o, \mathbf{s}_o) : \mathbb{R}^3 \times \Lambda \to S^3 \sim SU(2),$$

(11)

such that

$$S^3 \ni \pm 1 = \mathcal{A}(\mathbf{a}; \mathbf{e}_o, \mathbf{s}_o) = \begin{cases} \text{sign}\{ -\cos(\eta_{ae_o}) \} & \text{if } |\cos(\eta_{ae_o})| \geq -1 + \frac{2}{\sqrt{1 + 3 \left( \frac{\eta_{zs_o}}{\pi} \right)}} \\ 0 & \text{if } |\cos(\eta_{ae_o})| < -1 + \frac{2}{\sqrt{1 + 3 \left( \frac{\eta_{zs_o}}{\pi} \right)}} \end{cases},$$

(12)