Complete State from the Triangle Inequality for Quaternions:

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Recall that a parallelized 3-sphere, S^3 [or the group SU(2)], is a set of unit quaternions, defined as

$$S^3 \,:=\, \left\{ \mathbf{q}(\phi_o,\, I\cdot\mathbf{v}_o) \,:=\, \exp\left[\,(I\cdot\mathbf{v}_o)\,\phi_o\,\right] \;\middle|\; ||\,\mathbf{q}(\phi_o,\, I\cdot\mathbf{v}_o)\,|| \,=\, 1 \right\}, \tag{1}$$

where $I\cdot\mathbf{v}_o$ is an initial bivector rotating about $\mathbf{v}_o \in \mathrm{I\!R}^3$ and ϕ_o is half of its initial rotation angle. Using the notation of geometric product, $\mathbf{ab} \,=\, \mathbf{a}\cdot\mathbf{b} \,+\, \mathbf{a}\wedge\mathbf{b}$, consider now two quaternions within a 3-sphere, say $\mathbf{p}_o(\eta_{\mathbf{ne}_o},\, \mathbf{n}\wedge\mathbf{e}_o)$ and $\mathbf{q}_o(\eta_{\mathbf{zs}_o},\, \mathbf{z}\wedge\mathbf{s}_o)$, with $\mathbf{z}$ as a fixed reference vector, defined by

$$\mathbf{p}_o(\eta_{\mathbf{ne}_o},\, \mathbf{n}\wedge\mathbf{e}_o) \,:=\, \cos(\,\eta_{\mathbf{ne}_o}) \,+\, \frac{\mathbf{n}\wedge\mathbf{e}_o}{||\mathbf{n}\wedge\mathbf{e}_o||}\,\sin(\,\eta_{\mathbf{ne}_o}) \tag{2}$$

$$\text{and}\quad \mathbf{q}_o(\eta_{\mathbf{zs}_o},\, \mathbf{z}\wedge\mathbf{s}_o) \,:=\, \cos(\,\eta_{\mathbf{zs}_o}\,) \,+\, \frac{\mathbf{z}\wedge\mathbf{s}_o}{||\mathbf{z}\wedge\mathbf{s}_o||}\,\sin(\,\eta_{\mathbf{zs}_o}). \tag{3}$$

Note that, although $\mathbf{p}_o$ and $\mathbf{q}_o$ are normalized to unity, their sum $\mathbf{p}_o + \mathbf{q}_o$ need not be. In fact, from the triangle inequality (which holds for any arbitrary pair of quaternions) we have the inequality

$$||\mathbf{p}_o|| \,+\, ||\mathbf{q}_o|| \,\geq\, ||\mathbf{p}_o +\, \mathbf{q}_o||\,. \tag{4}$$

Multiplying with $||\mathbf{p}_o|| = 1$ on both sides, and simplifying a bit, reduces this triangle inequality to

$$||\mathbf{p}_o||^2 \,\geq\, ||\mathbf{p}_o +\, \mathbf{q}_o|| \,-\, 1\,. \tag{5}$$

This inequality allows us to make the following choice for the set of complete states for our system:

$$\Lambda \,:=\, \left\{ (\mathbf{p}_o,\, \mathbf{q}_o) \;\middle|\; ||\mathbf{p}_o + \mathbf{q}_o|| \,=\, 1 \,+\, \sin^2(\,\eta_{\mathbf{ne}_o}) \,+\, f^2(\,\eta_{\mathbf{zs}_o}) \quad \forall\,\mathbf{n}\in T_pS^3 \right\}, \tag{6}$$

where $f(\,\eta_{\mathbf{zs}_o})$ is an arbitrary function of $\eta_{\mathbf{zs}_o}$, which, as we shall soon see, satisfies the condition

$$f(\,\eta_{\mathbf{zs}_o}) \,\leq\, |\cos(\,\eta_{\mathbf{ne}_o})|. \tag{7}$$

Thus the choice (6) respects the condition $||\mathbf{p}_o +\, \mathbf{q}_o|| \,\leq\, 2$ following from Eq. (4). Substituting for

$$||\mathbf{p}_o||^2 \,=\, \cos^2(\,\eta_{\mathbf{ne}_o}) \,+\, \sin^2(\,\eta_{\mathbf{ne}_o}) \tag{8}$$

from Eq. (2), and for $||\mathbf{p}_o + \mathbf{q}_o||$, with $f(\,\eta_{\mathbf{zs}_o}) = -1 + \frac{2}{\sqrt{1+3\left(\frac{\eta_{\mathbf{zs}_o}}{\pi}\right)}}$ from Eq. (6), into Eq. (5) leads to

$$\boxed{\,|\cos(\,\eta_{\mathbf{ne}_o})| \,\geq\, -1 + \frac{2}{\sqrt{1+3\left(\frac{\eta_{\mathbf{zs}_o}}{\pi}\right)}}\,} \tag{9}$$

Without loss of generality we can now identify $\mathbf{e}_o \in \mathrm{I\!R}^3$ as a random vector and $\eta_{\mathbf{zs}_o} \equiv \eta_o \in [0,\, \pi]$ as a random scalar. This finally allows us to rewrite the set (6) of complete states of the system as

$$\Lambda \,:=\, \left\{ (\mathbf{p}_o,\, \mathbf{q}_o) \;\middle|\; |\cos(\,\eta_{\mathbf{ne}_o})| \,\geq\, -1 + \frac{2}{\sqrt{1+3\left(\frac{\eta_{\mathbf{zs}_o}}{\pi}\right)}} \quad \forall\,\mathbf{n}\in T_pS^3 \right\}, \tag{10}$$

which, for $\mathbf{n} = \mathbf{a}$, corresponds to measurement results of the form

$$\pm\,1 \,=\, \mathscr{A}(\mathbf{a};\, \mathbf{e}_o,\, \mathbf{s}_o) : \mathrm{I\!R}^3 \times \Lambda \longrightarrow S^3 \sim \mathrm{SU}(2)\,, \tag{11}$$

such that

$$S^3 \ni \pm\,1 \,=\, \mathscr{A}(\mathbf{a};\, \mathbf{e}_o,\, \mathbf{s}_o) \,=\, \begin{cases} \mathrm{sign}\{-\cos(\,\eta_{\mathbf{ae}_o}\,)\} & \text{if } \;|\cos(\,\eta_{\mathbf{ae}_o})| \,\geq\, -1 + \frac{2}{\sqrt{1+3\left(\frac{\eta_{\mathbf{zs}_o}}{\pi}\right)}} \\[2ex] 0 & \text{if } \;|\cos(\,\eta_{\mathbf{ae}_o})| \,<\, -1 + \frac{2}{\sqrt{1+3\left(\frac{\eta_{\mathbf{zs}_o}}{\pi}\right)}}. \end{cases} \tag{12}$$